Statistical Natural Language Processing

A refresher on probability theory

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Why probability theory?

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Short answer: practice proved otherwise.

Slightly long answer

- Many linguistic phenomena are better explained as tendencies, rather than fixed rules
- Probability theory captures many characteristics of (human) cognition, language is not an exception

What is probability?

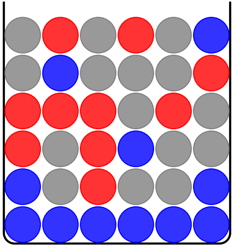
- Probability is a measure of (un)certainty
- We quantify the probability of an event with a number between 0 and 1
  - 0 the event is impossible
  - 0.5 the event is as likely to happen as it is not
  - 1 the event is certain
- The set of all possible *outcome*s of a trial is called *sample space* (Ω)
- An *event* (E) is a set of outcomes

Axioms of probability state that

1. $P(E) \in \mathbb{R}, P(E) \geqslant 0$
2. $P(\Omega) = 1$
3. For *disjoint* events E_1 and E_2, $P(E_1 \cup E_2) = P(E_1) + P(E_2)$

What you should already know

$P(\color{blue}\bullet) = ?$



- $P(\{\color{blue}\bullet\}) = 4/9$
- $P(\{\color{red}\bullet\}) = 4/9$
- $P(\{\color{gray}\bullet\}) = 1/9$
- $P(\{\color{blue}\bullet\color{black}, \color{red}\bullet\}) = 8/9$
- $P(\{\color{blue}\bullet\color{black}, \color{red}\bullet\color{black}, \color{gray}\bullet\}) = 1$

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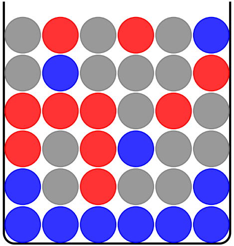

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- $P(\{{\color{blue}\bullet}, {\color{red}\bullet}\}) = 8/9$
- $P(\{{\color{blue}\bullet}, {\color{red}\bullet}, {\color{gray}\bullet}\}) = 1$
- $P(\{{\color{blue}\bullet}{\color{blue}\bullet}\}) = 16/81$
- $P(\{{\color{red}\bullet}{\color{blue}\bullet}\}) = 16/81$
- $P(\{{\color{gray}\bullet}{\color{blue}\bullet}\}) = 4/81$
- $P(\{{\color{gray}\bullet}{\color{gray}\bullet}\}) = 1/81$
- $P(\{{\color{blue}\bullet}{\color{red}\bullet}, {\color{gray}\bullet}{\color{red}\bullet}\}) = 20/81$

Where do probabilities come from

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Two major (rival) ways of assigning probabilities to events are

- Frequentist (objective) probabilities: probability of an event is its relative frequency (in the limit)
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Random variables

- A random variable is a variable whose value is subject to uncertainties
- A random variable is always a number
- Think of a random variable as mapping between the outcomes of a trial to (a vector of) real numbers (a real valued function on the sample space)
- Example outcomes of uncertain experiments
  - height or weight of a person
  - length of a word randomly chosen from a corpus
  - whether an email is spam or not
  - the first word of a book, or first word uttered by a baby

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  Note: not all of these are numbers

Random variables

mapping outcomes to real numbers

- Continuous
  - frequency of a sound signal: 100.5, 220.3, 4321.3 …
- Discrete
  - Number of words in a sentence: 2, 5, 10, …
  - Whether a review is negative or positive:

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Outcome	Noun	Verb	Adj	Adv	…

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Outcome	Noun	Verb	Adj	Adv	…
Value	1	2	3	4	…

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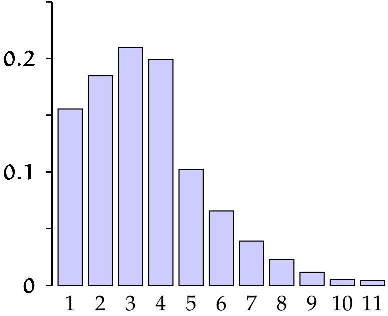
  - The POS tag of a word:

Outcome	Noun	Verb	Adj	Adv	…
Value					
…or	1 0 0 0 0	0 1 0 0 0	0 0 1 0 0	0 0 0 1 0	…

Probability mass function

Example: probabilities for sentence length in words

- *Probability mass function* (*PMF*) of a *discrete* random variable (X) maps every possible (x) value to its probability ($P(X = x)$).



x	$P(X = x)$
1	0.155
2	0.185
3	0.210
4	0.194
5	0.102
6	0.066
7	0.039
8	0.023
9	0.012
10	0.005
11	0.004

Populations, distributions, samples

- In many applications, we use probability theory to make inferences about a possibly infinite *population*
- A probability distribution is a way to characterize a population
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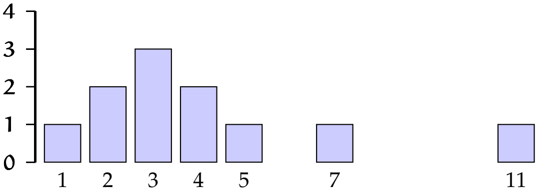
A sample from the distribution in the previous slide:

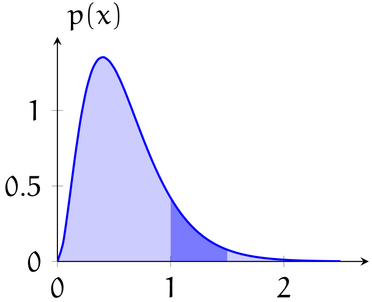
$$[1, 2, 2, 3, 3, 3, 4, 4, 5, 7, 11]$$



Probability density function (PDF)

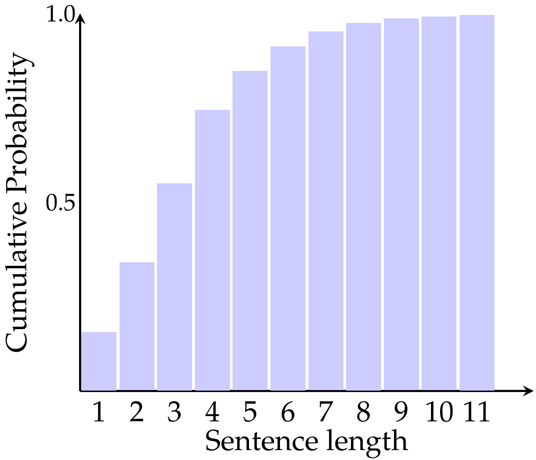
- Continuous variables have *probability density functions*
- $p(x)$ is not a probability (note the notation: we use lowercase p for PDF)
- Area under $p(x)$ sums to 1
- $P(X = x) = 0$
- Non zero probabilities are possible for ranges:

$$P(a \leqslant x \leqslant b) = \int_a^b p(x)dx$$



Cumulative distribution function

- $F_X(x) = P(X \leqslant x)$



Length	Prob.	C. Prob.
1	0.16	0.16
2	0.18	0.34
3	0.21	0.55
4	0.19	0.74
5	0.10	0.85
6	0.07	0.91
7	0.04	0.95
8	0.02	0.97
9	0.01	0.99
10	0.01	0.99
11	0.00	1.00

Expected value

- Expected value (mean) of a random variable X is,

$$E[X] = \mu = \sum_{i=1}^{n} P(x_i)x_i = P(x_1)x_1 + P(x_2)x_2 + \ldots + P(x_n)x_n$$

- More generally, expected value of a function of X is

$$E[f(X)] = \sum_{x} P(x)f(x)$$

- Expected value is a measure of central tendency
- Note: it is not the 'most likely' value
- Expected value is linear

$$E[aX + bY] = aE[X] + bE[Y]$$

Variance and standard deviation

- Variance of a random variable X is,

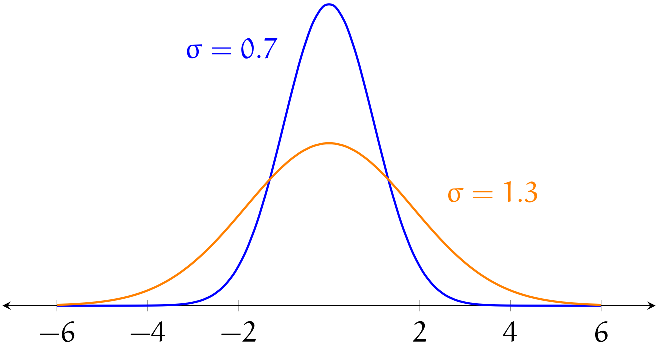
$$Var(X) = \sigma^2 = \sum_{i=1}^{n} P(x_i)(x_i - \mu)^2 = E[X^2] - (E[X])^2$$

- It is a measure of spread, divergence from the central tendency
- The square root of variance is called standard deviation

$$\sigma = \sqrt{\left(\sum_{i=1}^{n} P(x_i)x_i^2\right) - \mu^2}$$

- Standard deviation is in the same units as the values of the random variable
- Variance is not linear: $\sigma^2_{X+Y} \neq \sigma^2_X + \sigma^2_Y$ (neither the σ)

Example: two distributions with different variances



Short divergence: Chebyshev's inequality

For any probability distribution, and $k > 1$,

$$P(|x - \mu| > k\sigma) \leqslant \frac{1}{k^2}$$

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Distance from μ	2σ	3σ	5σ	10σ	100σ
Probability	0.25	0.11	0.04	0.01	0.0001

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For any probability distribution, and $k > 1$,

$$P(|x - \mu| > k\sigma) \leqslant \frac{1}{k^2}$$

Distance from μ	2σ	3σ	5σ	10σ	100σ
Probability	0.25	0.11	0.04	0.01	0.0001

This also shows why standardizing values of random variables,

$$z = \frac{x - \mu}{\sigma}$$

makes sense (the normalized quantity is often called the z-score).

Median and mode of a random variable

Median is the mid-point of a distribution. Median of a random variable is defined as the number m that satisfies

$$P(X \leqslant m) \geqslant \frac{1}{2} \quad \text{and} \quad P(X \geqslant m) \geqslant \frac{1}{2}$$

- Median of $1, 4, 5, 8, 10$ is 5
- Median of $1, 4, 5, 7, 8, 10$ is 6

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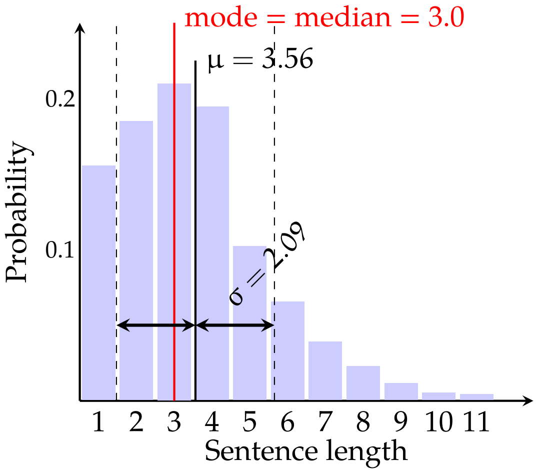
- Median of $1, 4, 5, 7, 8, 10$ is 6

Mode is the value that occurs most often in the data.

- Modes appear as peaks in probability mass (or density) functions
- Mode of $1, 4, 4, 8, 10$ is 4
- Modes of $1, 4, 4, 8, 9, 9$ are 4 and 9

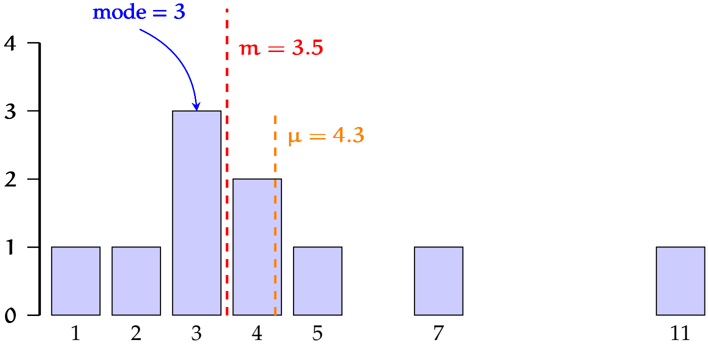
Mode, median, mean, standard deviation

Visualization on sentence length example

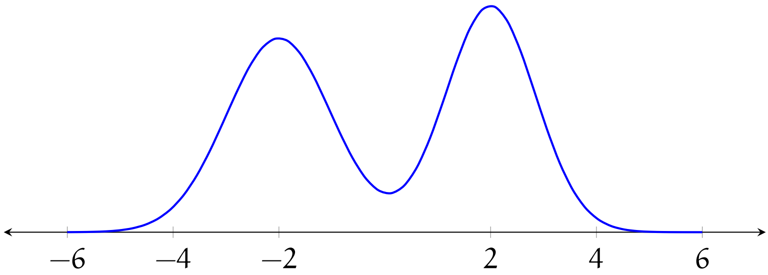


Mode, median, mean

sensitivity to extreme values



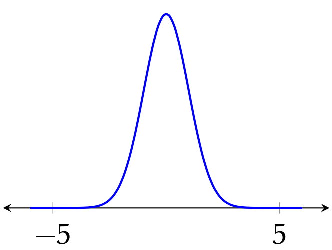
Multimodal distributions

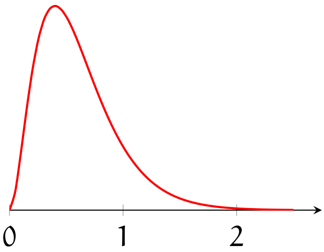


- A distribution is multimodal if it has multiple modes
- Multimodal distributions often indicate confounding variables

Skew

- Another important property of a probability distribution is its *skew*
- symmetric distributions have no skew
- positively skewed distributions have a long *tail* on the right
- negatively skewed distributions have a long left tail



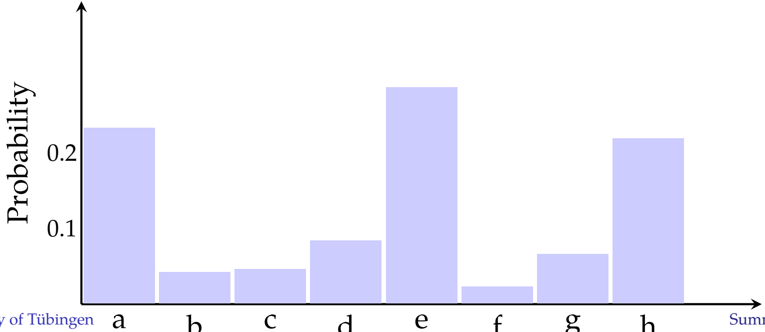


Another example

A probability distribution over letters

- We have a hypothetical language with 8 letters with the following probabilities

Lett.	a	b	c	d	e	f	g	h
Prob.	0.23	0.04	0.05	0.08	0.29	0.02	0.07	0.22



Probability distributions

- Some random variables (approximately) follow a distribution that can be parametrized with a number of parameters
- For example, Gaussian (or normal) distribution is conventionally parametrized by its mean (μ) and variance (σ^2)
- Common notation we use for indicating that a variable X follows a particular distribution is

$$X \sim \mathrm{Normal}(\mu, \sigma^2) \quad \text{or} \quad X \sim \mathcal{N}(\mu, \sigma^2).$$

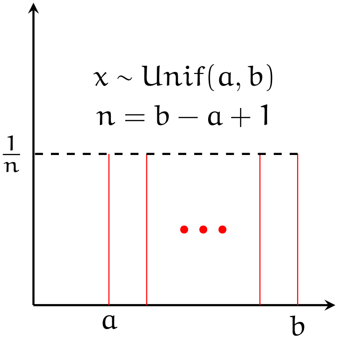
- For the rest of this lecture, we will revise some of the important probability distributions

Probability distributions (cont)

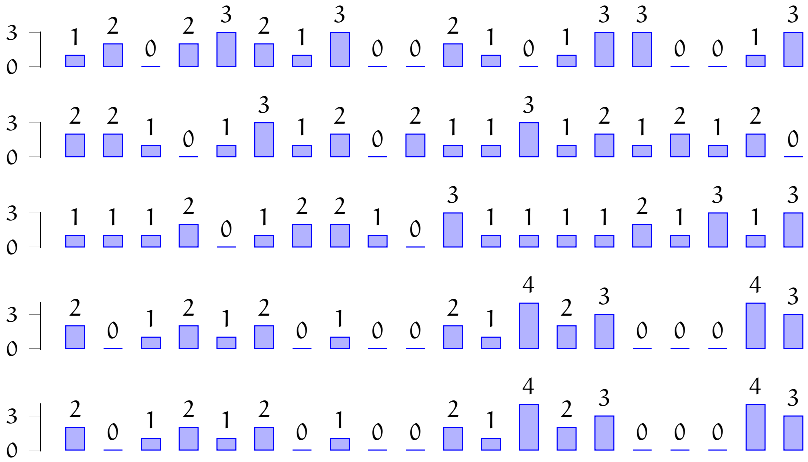
- A probability distribution is called *univariate* if it was defined on scalars
- *multivariate* probability distributions are defined on vectors
- Probability distributions are abstract mathematical objects (functions that map events/outcomes to probabilities)
- In real life, we often deal with *samples*
- A probability distribution is generative device: it can generate samples
- Finding most likely probability distribution from a sample is called *inference* (next week)

Uniform distribution (discrete)

- A uniform distribution assigns equal probabilities to all values in range $[a, b]$, where a and b are the parameters of the distribution
- Probabilities of the values outside range is 0
- $\mu = \frac{b+a}{2}$
- $\sigma_2 = \frac{(b-a+1)^2-1}{12}$
- There is also an analogous continuous uniform distribution



Samples from a uniform distribution



Bernoulli distribution

Bernoulli distribution characterizes simple random experiments with two outcomes

- Coin flip: heads or tails
- Spam detection: spam or not
- Predicting gender: female or male

We denote (arbitrarily) one of the possible values with 1 (often called a success), the other with 0 (often called a failure)

$$
\begin{aligned}
P(X = 1) &= p \\
P(X = 0) &= 1 - p \\
P(X = k) &= p^k (1 - p)^{1-k} \\
\mu_X &= p \\
\sigma_X^2 &= p(1 - p)
\end{aligned}
$$

Binomial distribution

Binomial distribution is a generalization of Bernoulli distribution to n trials, the value of the random variable is the number of 'successes' in the experiment

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$
$$\mu_X = np$$
$$\sigma_X^2 = np(1-p)$$

Remember that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Categorical distribution

- Extension of Bernoulli to k mutually exclusive outcomes
- For any k-way event, distribution is parametrized by k parameters $p_1, \ldots, p_k$ ($k - 1$ independent parameters) where

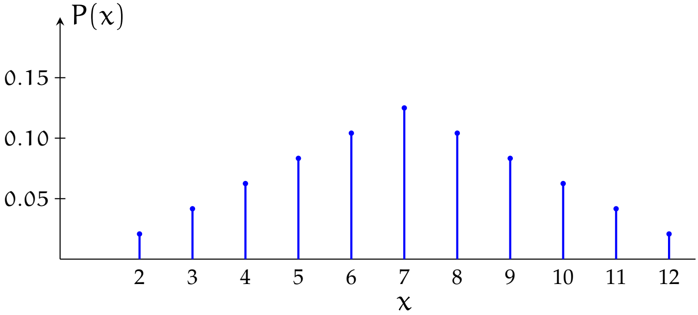
$$\sum_{i=1}^{k} p_i = 1$$

$$E[x_i] = p_i$$
$$Var(x_i) = p_i(1 - p_i)$$

- Similar to Bernoulli–binomial generalization, *multinomial* distribution is the generalization of categorical distribution to n trials

Categorical distribution example

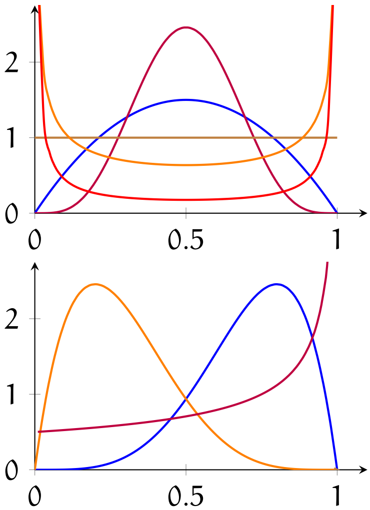
sum of the outcomes from roll of two fair dice



Beta distribution

- Beta distribution is defined in range $[0, 1]$
- It is characterized by two parameters α and β

$$p(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}}$$



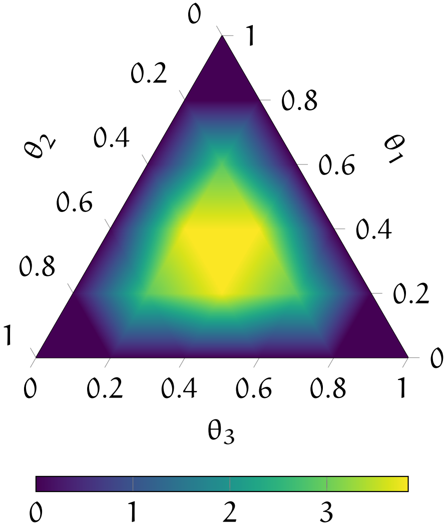
Beta distribution

where do we use it

- A common use is the random variables whose values are probabilities
- Particularly important in Bayesian methods as a conjugate prior of Bernoulli and Binomial distributions
- *Dirichlet distribution* generalizes Beta to k-dimensional vectors whose components are in range $(0, 1)$ and $\|x\|_1 = 1$.
- Dirichlet distribution is also used often in NLP, e.g., *latent Dirichlet allocation* is a well know method for topic modeling

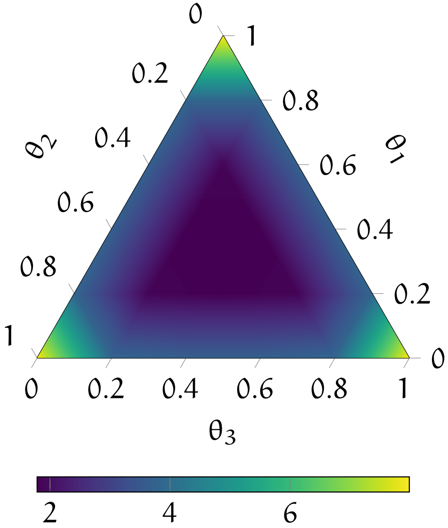
Example Dirichlet distributions

$\theta = (2, 2, 2)$



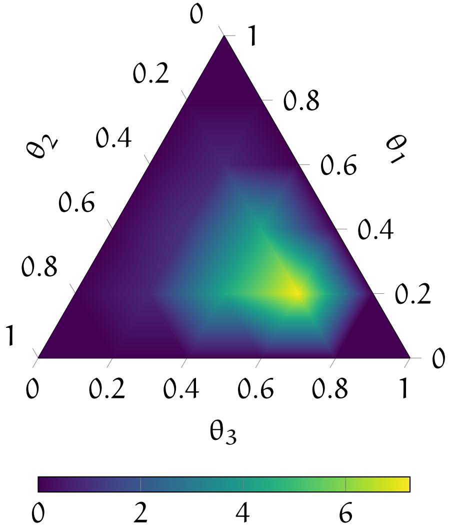
Example Dirichlet distributions

$\theta = (0.8, 0.8, 0.8)$



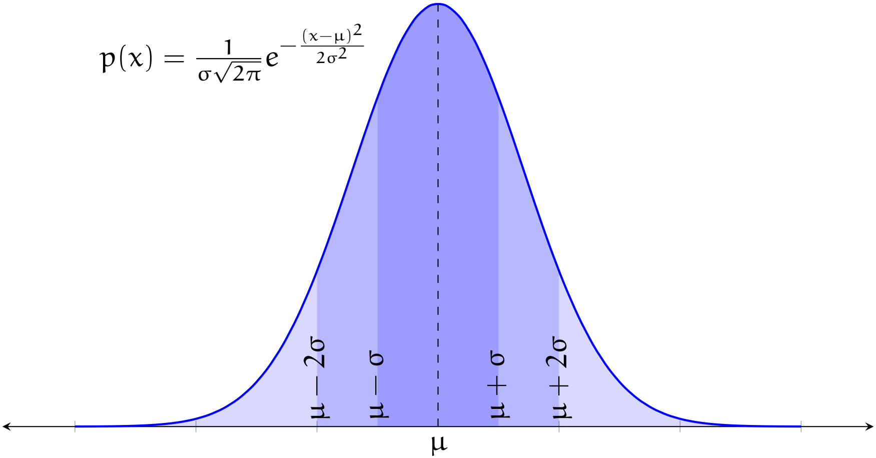
Example Dirichlet distributions

$\theta = (2, 2, 4)$



Gaussian (normal) distribution

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



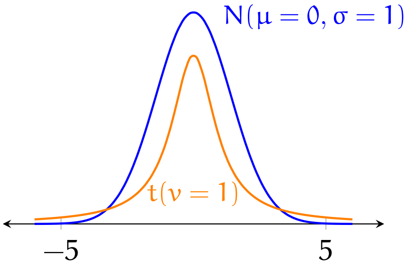
Short detour: central limit theorem

Central limit theorem (CLT) states that the sum of a large number of independent and identically distributed variables (i.i.d.) is normally distributed.

- Expected value (average) of means of samples from any distribution will be distributed normally
- Many (inference) methods in statistics and machine learning work because of this fact

Student's t-distribution

- T-distribution is another important distribution
- It is similar to normal distribution, but it has heavier tails
- It has one parameter: *degree of freedom* (ν)



Joint and marginal probability

Two random variables form a *joint probability distribution*.

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Two random variables form a *joint probability distribution*.

An example: consider the letter bigrams.

	a	**b**	**c**	**d**	**e**	**f**	**g**	**h**
a	0.04	0.02	0.02	0.03	0.05	0.01	0.02	0.06
b	0.01	0.00	0.00	0.00	0.01	0.00	0.00	0.01
c	0.02	0.00	0.00	0.00	0.01	0.00	0.00	0.01
d	0.02	0.00	0.00	0.01	0.02	0.00	0.01	0.02
e	0.06	0.02	0.01	0.03	0.08	0.01	0.01	0.07
f	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.01
g	0.01	0.00	0.00	0.01	0.02	0.00	0.01	0.02
h	0.08	0.00	0.00	0.01	0.10	0.00	0.01	0.02

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c	0.02	0.00	0.00	0.00	0.01	0.00	0.00	0.01	0.05
d	0.02	0.00	0.00	0.01	0.02	0.00	0.01	0.02	0.08
e	0.06	0.02	0.01	0.03	0.08	0.01	0.01	0.07	0.29
f	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.01	0.02
g	0.01	0.00	0.00	0.01	0.02	0.00	0.01	0.02	0.07
h	0.08	0.00	0.00	0.01	0.10	0.00	0.01	0.02	0.22
	0.23	0.04	0.05	0.08	0.29	0.02	0.07	0.22	

Expected values of joint distributions

$$E[f(X,Y)] = \sum_x \sum_y P(x,y) f(x,y)$$

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$$\mu_X = E[X] = \sum_{x} \sum_{y} P(x, y) x$$

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Expected values of joint distributions

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$$\mu_Y = E[Y] = \sum_{x} \sum_{y} P(x, y) y$$

We can simplify the notation by vector notation, for $\boldsymbol{\mu} = (\mu_x, \mu_y)$,

$$\boldsymbol{\mu} = \sum_{\boldsymbol{x} \in XY} \boldsymbol{x} P(\boldsymbol{x})$$

where vector $\boldsymbol{x}$ ranges over all possible combinations of the values of random variables X and Y.

Variances of joint distributions

$$\sigma_X^2 = \sum_x \sum_y P(x, y)(x - \mu_X)^2$$

$$\sigma_Y^2 = \sum_x \sum_y P(x, y)(y - \mu_Y)^2$$

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Again, using vector/matrix notation we can define the *covariance matrix* ($\boldsymbol{\Sigma}$) as

$$\boldsymbol{\Sigma} = E[(\mathbf{x} - \boldsymbol{\mu})^2]$$

Covariance and the covariance matrix

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{YX} & \sigma_Y^2 \end{bmatrix}$$

- The main diagonal of the covariance matrix contains the variances of the individual variables
- Non-diagonal entries are the covariances of the corresponding variables
- Covariance matrix is symmetric ($\sigma_{XY} = \sigma_{YX}$)
- For a joint distribution of k variables we have a covariance matrix of size $k \times k$

Correlation

Correlation is a normalized version of covariance

$$r = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

Correlation coefficient (r) takes values between -1 and 1

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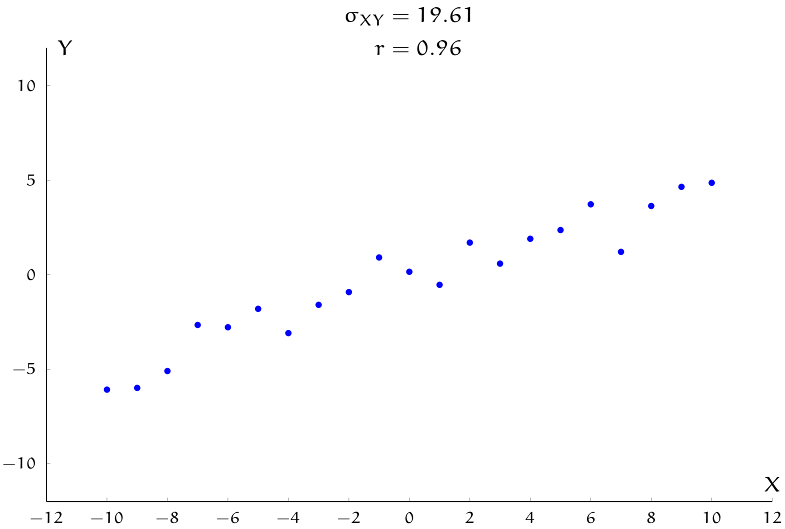
Correlation coefficient (r) takes values between -1 and 1

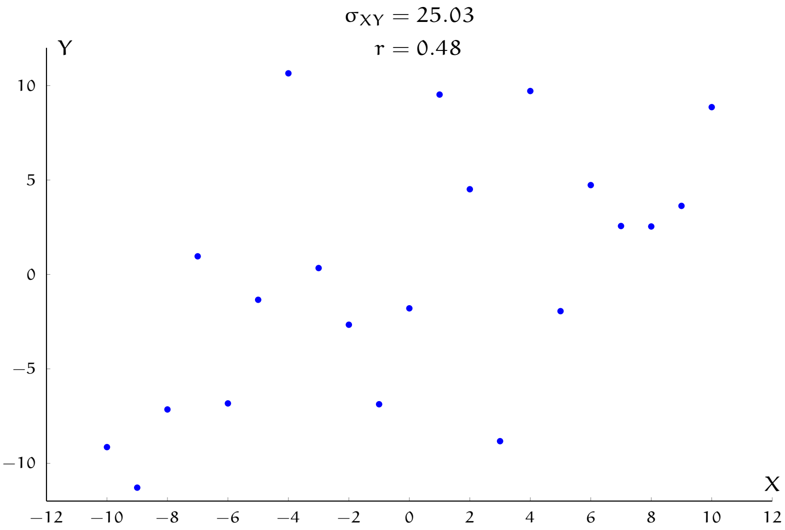
- 1 Perfect positive correlation.
- $(0, 1)$ positive correlation: x increases as y increases.
- 0 No correlation, variables are independent.
- $(-1, 0)$ negative correlation: x decreases as y increases.
- -1 Perfect negative correlation.

Note: like covariance, correlation is a symmetric measure.

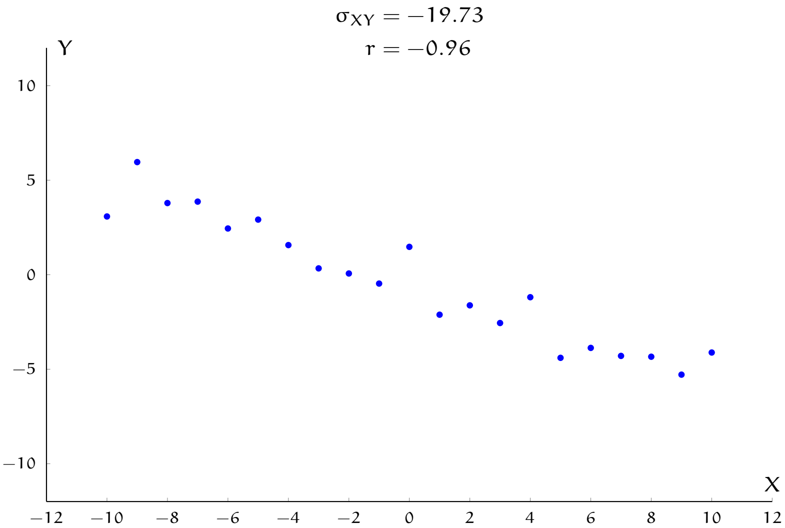
Correlation: visualization (1)



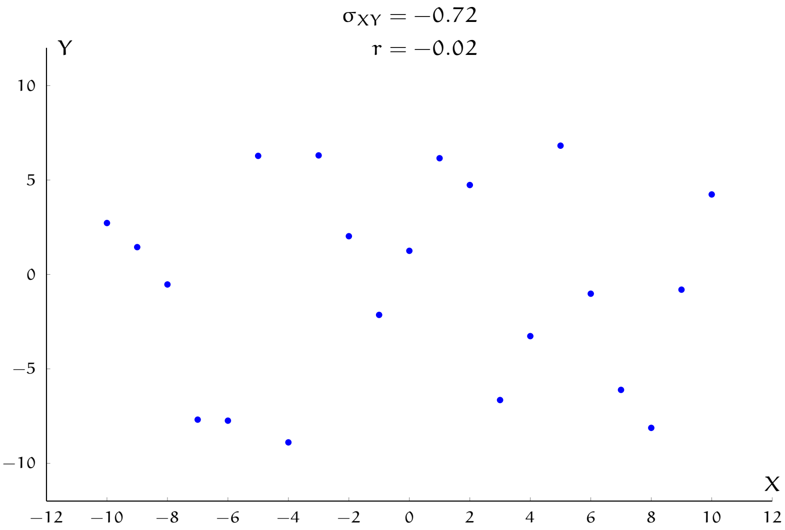
Correlation: visualization (2)



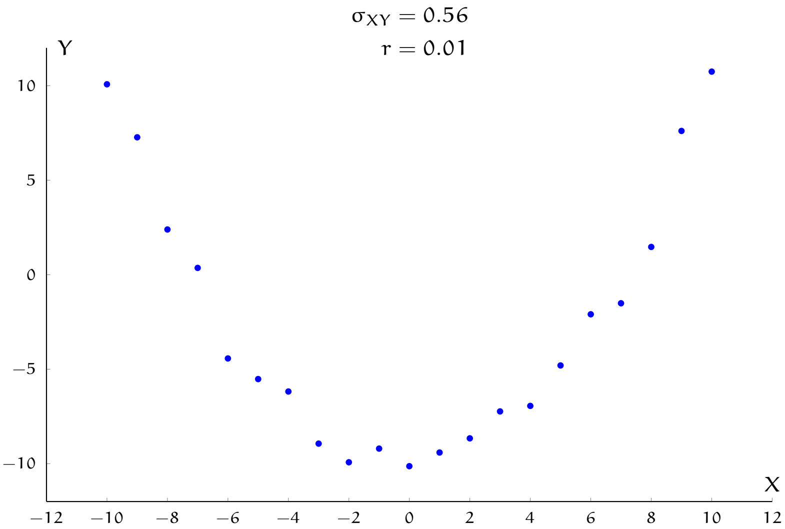
Correlation: visualization (3)



Correlation: visualization (4)



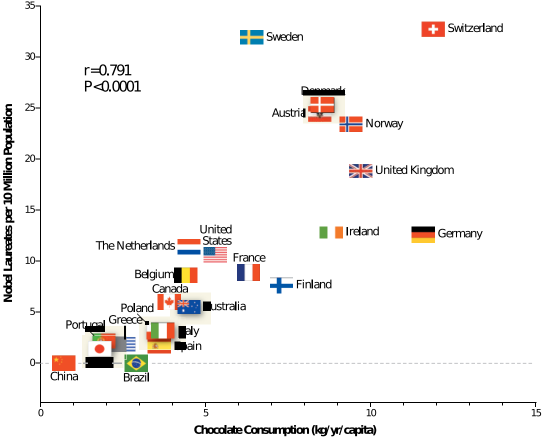
Correlation: visualization (5)



Correlation and independence

- Statistical (in)dependence is an important concept (in ML)
- The correlation (or covariance) of independent random variables is 0
- The reverse is not true: 0 correlation does not imply independence
- Correlation measures a linear dependence (relationship) between two variables, non-linear dependences are not measured by correlation

Short divergence: correlation and causation



From **messerli2012**.

Conditional probability

In our letter bigram example, given that we know that the first letter is e, what is the probability of second letter being d?

	a	b	c	d	e	f	g	h	
a	0.04	0.02	0.02	0.03	0.05	0.01	0.02	0.06	0.23
b	0.01	0.00	0.00	0.00	0.01	0.00	0.00	0.01	0.04
c	0.02	0.00	0.00	0.00	0.01	0.00	0.00	0.01	0.05
d	0.02	0.00	0.00	0.01	0.02	0.00	0.01	0.02	0.08
e	0.06	0.02	0.01	0.03	0.08	0.01	0.01	0.07	0.29
f	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.01	0.02
g	0.01	0.00	0.00	0.01	0.02	0.00	0.01	0.02	0.07
h	0.08	0.00	0.00	0.01	0.10	0.00	0.01	0.02	0.22
	0.23	0.04	0.05	0.08	0.29	0.02	0.07	0.22	

$P(L_1 = e, L_2 = d) = 0.025940365$

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d	0.02	0.00	0.00	0.01	0.02	0.00	0.01	0.02	0.08
e	0.06	0.02	0.01	0.03	0.08	0.01	0.01	0.07	0.29
f	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.01	0.02
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$P(L_1 = e, L_2 = d) = 0.025940365$ $\qquad$ $P(L_1 = e) = 0.28605090$

$$P(L_2 = d | L_1 = e) = \frac{P(L_1 = e, L_2 = d)}{P(L_1 = e)}$$

Conditional probability (2)

In terms of probability mass (or density) functions,

$$P(X|Y) = \frac{P(X,Y)}{P(Y)}$$

If two variables are independent, knowing the outcome of one does not affect the probability of the other variable:

$$P(X|Y) = P(X) \qquad P(X,Y) = P(X)P(Y)$$

More notes on notation/interpretation:

$P(X = x, Y = y)$ Probability that $X = x$ and $Y = y$ at the same time (joint probability)

$P(Y = y)$ Probability of $Y = y$, for any value of X ($\sum_{x \in X} P(X = x, Y = y)$) (marginal probability)

$P(X = x|Y = y)$ Knowing that we $Y = y$, $P(X = x)$ (conditional probability)

Bayes' rule

$$P(X|Y) = \frac{P(Y|X)P(X)}{P(Y)}$$

- This is a direct result of the axioms of the probability theory
- It is often useful as it 'inverts' the conditional probabilities
- The term $P(X)$, is called prior
- The term $P(Y|X)$, is called likelihood
- The term $P(X|Y)$, is called posterior

Example application of Bayes' rule

We use a test t to determine whether a patient has condition/illness c

- If a patient has c test is positive 99% of the time: $P(t|c) = 0.99$

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$$P(c|t) = \frac{P(t|c)P(c)}{P(t)} = \frac{P(t|c)P(c)}{P(t|c)P(c) + P(t|\neg c)P(\neg c)} = 0.001$$

Chain rule

We rewrite the relation between the joint and the conditional probability as

$$P(X, Y) = P(X|Y)P(Y)$$

We can also write the same quantity as,

$$P(X, Y) = P(Y|X)P(X)$$

For more than two variables, one can write

$$P(X, Y, Z) = P(Z|X, Y)P(Y|X)P(X) = P(X|Y, Z)P(Y|Z)P(Z) = \ldots$$

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In general, for any number of random variables, we can write

$$P(X_1, X_2, \ldots, X_n) = P(X_1|X_2, \ldots, X_n)P(X_2, \ldots, X_n)$$

Conditional independence

If two random variables are conditionally independent:

$$P(X, Y|Z) = P(X|Z)P(Y|Z)$$

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This is often used for simplifying the statistical models. For example in spam filtering with Naive Bayes classifier, we are interested in

$$P(w_1, w_2, w_3|\text{spam}) =$$
$$P(w_1|w_2, w_3, \text{spam})P(w_2|w_3, \text{spam})P(w_3|\text{spam})$$

with the assumption that occurrences of words are independent of each other given we know the email is spam or not,

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Continuous random variables

some reminders

The rules and quantities we discussed above apply to continuous random variables with some differences

- For continuous variables, $P(X = x) = 0$
- We cannot talk about probability of the variable being equal to a single real number
- But we can define probabilities of ranges
- For all formulas we have seen so far, replace summation with integrals
- Probability of a range:

$$P(a < X < b) = \int_a^b p(x)dx$$

Multivariate continuous random variables

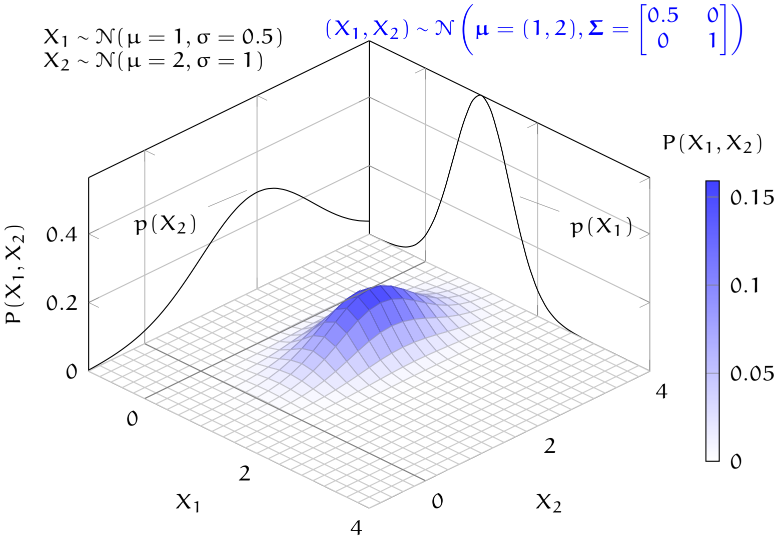
- Joint probability density

$$p(X, Y) = p(X|Y)p(Y) = p(Y|X)p(X)$$

- Marginal probability

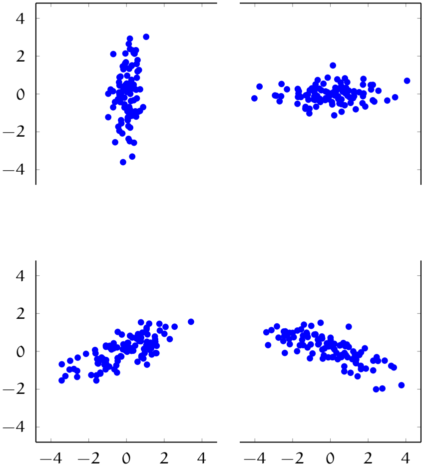
$$P(X) = \int_{-\infty}^{\infty} p(x, y)dy$$

Multivariate Gaussian distribution



Samples from bi-variate normal distributions

$$\Sigma = \begin{bmatrix} 0.5 & 0 \\ 0 & 2 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}$$



$$\Sigma = \begin{bmatrix} 0.5 & 0.7 \\ 0.7 & 2 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 2 & -0.7 \\ -0.7 & 0.5 \end{bmatrix}$$

Summary: some keywords

- Probability, sample space, outcome, event
- Outcome, event, sample space
- Random variables: discrete and continuous
- Probability mass function
- Probability density function
- Cumulative distribution function
- Expected value
- Variance / standard deviation
- Median and mode
- Skewness of a distribution
- Joint and marginal probabilities
- Covariance, correlation
- Conditional probability
- Bayes' rule
- Chain rule
- Some well-known probability distributions:

Bernoulli	binomial
categorical	multinomial
beta	Dirichlet
Gaussian	Student's t

Next

Fri	(now) Information theory
Mon	ML Intro / regression
Wed	First lab session

References and further reading

- **mackay2003** covers most of the topics discussed in a way quite relevant to machine learning. The complete book is available freely online (see the link below)
- See **grinstead2012** a more conventional introduction to probability theory. This book is also freely available
- For an influential, but not quite conventional approach, see **jaynes2007**